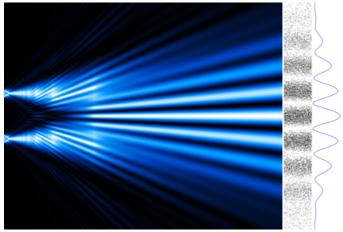
Peyresq 2021.

PART 1.

Wave-particle complementarity, welcher-weg information, Q eraser, entanglement, decoherence and all that.

Thomas Durt.

Einstein-Bohr debate about complementarity. 

- In 1927 at the Solvay conference Einstein discusses the double-slit experiment with Bohr.
- He proposes to equip the screen in which slits are located in order to measure the recoil occuring when the particle passes through one slit.
- This would make it possible to measure through which slit the particle passes (location) without destroying interferences.

Einstein-Bohr debate about complementarity.

- This would contradict Bohr's ideas according to which **either** we know where the particle is located (particle behaviour) **or** we measure interferences (wave behaviour) but then we ignore through which slit the particle passes.

- This is an aspect of Bohr's complementarity, illustrated by Heisenberg uncertainties (where δx is related to particle behaviour and δp_x to wave behaviour).

Einstein-Bohr debate about complementarity.

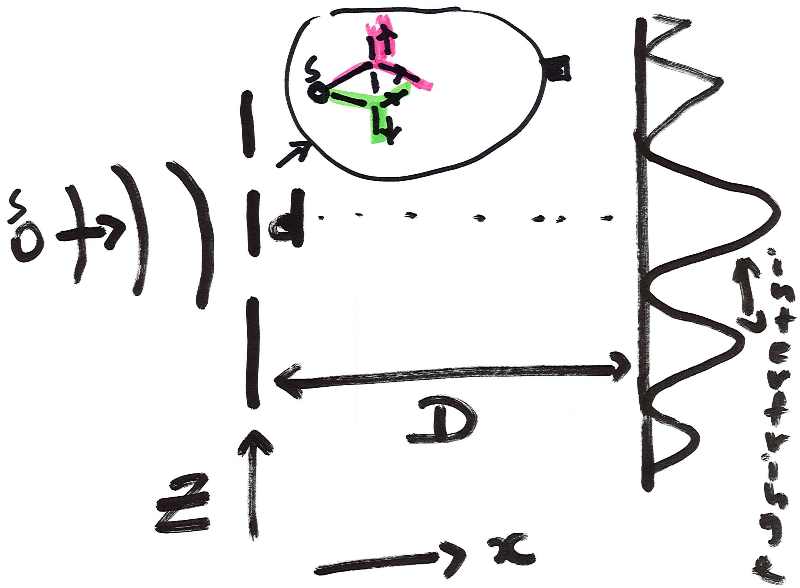
- Bohr realizes that if we impose to the screen to be a quantum object satisfying Heisenberg uncertainties, the uncertainty on the location of the slits is necessarily larger than the distance between fringes on the final screen (see figure netx page).

- Let us give here a sketchy and handwaving proof (for details see e.g. Examination of wave-particle duality via two-slit interference by Mario Rabinowitz https://arxiv.org/abs/physics/0302062), the distance between fringes is of the order of $\lambda \cdot (D/2d)$ where D is the distance between the screen with the slits and the final screen; now the recoil of the screen with the fringes is of the order of $(\hbar/\lambda) \cdot (d/D)$ where $\hbar/\lambda \approx p_x$ is the momentum of the incoming particle.

Einstein-Bohr debate about complementarity.

- If we want that the screen reveals through which slit the particle passes we must impose that δP_Z the uncertainty on the momentum of the screen is smaller than the recoil:

  $\delta P_Z < (\hbar/\lambda) \cdot (d/D)$.



Einstein-Bohr debate about complementarity.

- Making use of Heisenberg uncertainty applied to the screen vertical position Z we get $\delta Z \geq \hbar/2\delta P_Z$ so that, finally,
  $\delta Z > \lambda \cdot (D/2d)$
- Now, the distance between fringes is precisely $\lambda \cdot (D/2d)$.
- This means that if we are able to measure through which slit the particle passes, interferences at the level of the final fringe are averaged out over a distance larger than the interfringe and disappear.

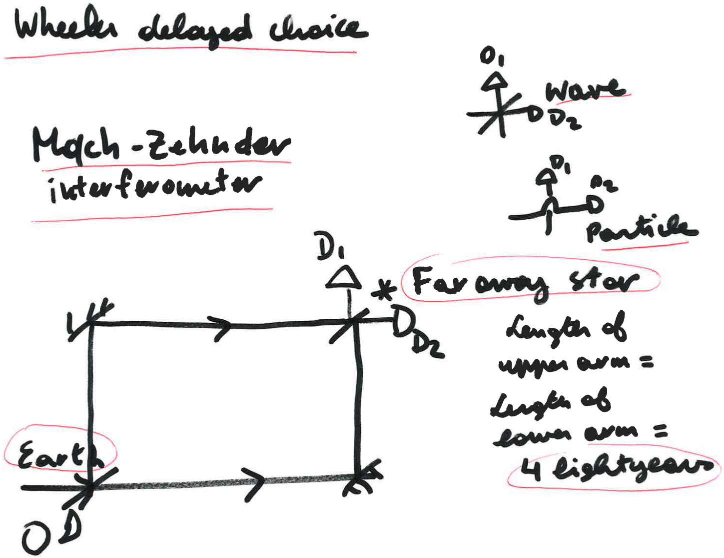
Einstein-Bohr debate about complementarity.

Apparently, the last word belongs to Bohr in this debate but this is not the end of the story as we shall discuss now.

- The reasoning of Bohr and Einstein presupposes that particles are localized at each time and follow continuous in time trajectories; the “mechanical” interaction with the screen is also not fully convincing.
- As Wheeler will show, in a delayed choice experiment, the wave or corpuscular nature of the quantum object is not fixed from the beginning, it can be imposed A POSTERIORI, which is a problem regarding causality.

Wheeler delayed choice experiment.

- Let us transpose Einstein-Bohr gedanken experiment to a "galactic" Mach-Zehnder device where light reaches the detectors after a travel of say 4 years.
- If (A) we put a beamsplitter at the end, interference fringes will appear and the behaviour is a wave behaviour.
- If (B) we do not put the beamsplitter, we know through which arm light passed and the behaviour is a corpuscle behaviour.



Wheeler delayed choice experiment.

- If we delay the choice between both possibilities A and B till the very end, we impose a posteriori that light behave as a wave or a particle, already when it enters the interferometer, 4 years before the choice.
- If the wave or particle nature of light is a "real" property of light, this property can be influenced from the future which sounds weird[a].
- This constitutes Wheeler's delayed choice paradox...

[a] In the literature, this is sometimes called Wheeler necromancy....

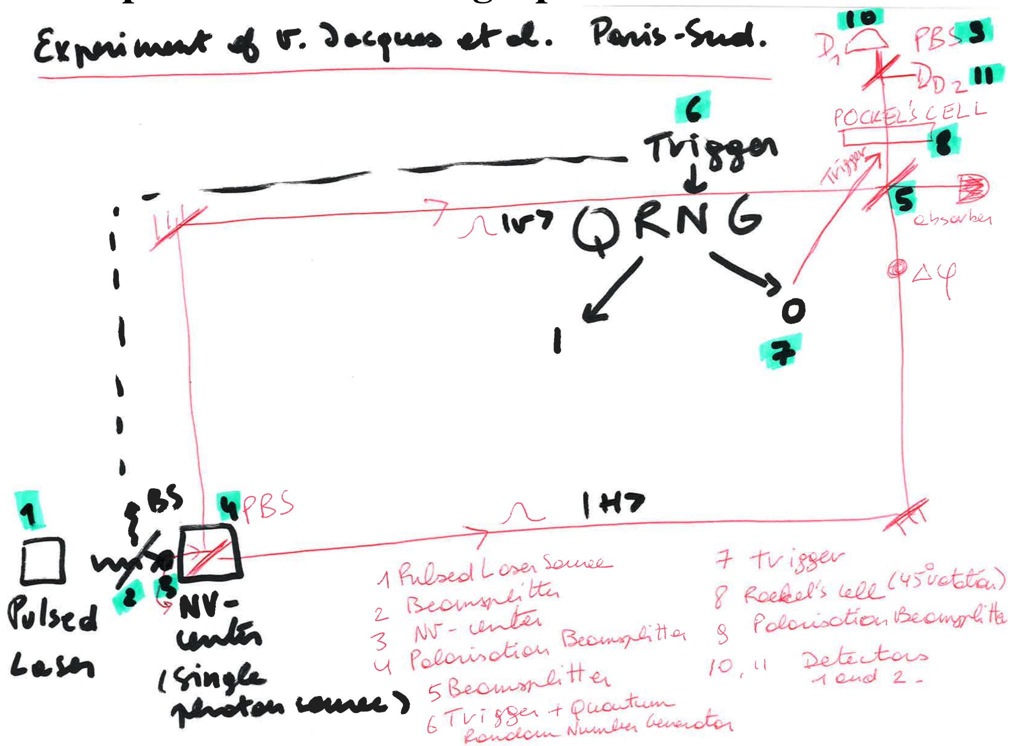
Wheeler delayed choice, an important experiment.

An important "recent" experiment (see next slide):

- Vincent Jacques et al.: **realized the delayed choice experiment with SINGLE PHOTON PULSES** for the 1st time...

  Vincent Jacques, E. Wu, Frederic Grosshans, Francois Treussart, Philippe Grangier, Alain Aspect and Jean-Francois Roche, Experimental realization of Wheeler's delayed-choice GedankenExperiment. Science, American Association for the Advancement of Science, 2007, 315 (5814), pp.966. 10.1126/science.1136303 . hal-00110392

Wheeler experiment with a single photon source...



Wheeler experiment with a single photon source...

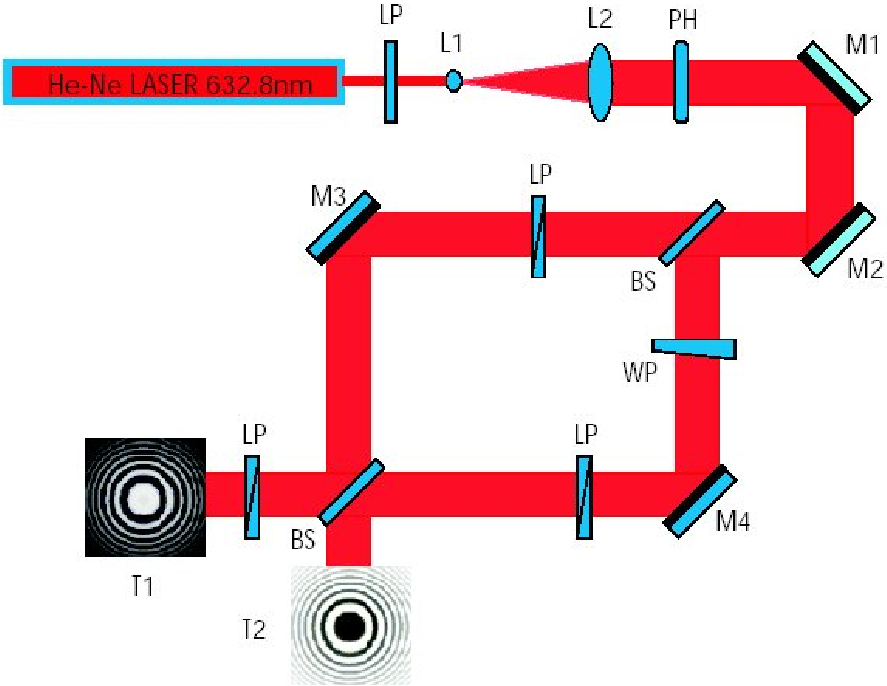
- A laser pulse produced in 1 gets splitted in two components at 2.
- One of these components triggers a nv-center which produces a single photon with diagonal polarization in 3.
- This photon gets splitted at 4 into a horizontal polarization (lower arm) and a vertical polarization (upper arm).
- The two components are recombined at 5.
- In the meanwhile a part of the laser pulse reaches at 6 a trigger which activates an independent QRNG (quantum randomnumber generator); this QRNG produces either a bit 1 or a bit 0 with probability 50 percent.
- If the bit 1 is produced nothing happens; if the bit 0 is produced, a trigger activates a pockel's cell which rotates by 45 degrees the polarisation at the vertical output of a beamsplitter (5).
- This means that before reaching the polarisation beamsplitter in 9 the polarisation state of the light is
  -either $\frac{1}{\sqrt{2}}(|Hor. > + e^{i\delta\phi}|Vert. >)$
  -or $\frac{1}{2}((1+e^{i\delta\phi})|Hor. > + (1-e^{i\delta\phi})|Vert. >)$

Wheeler experiment with a single photon source...

- If before reaching the polarisation beamsplitter in 9 the polarisation state of the light is $\frac{1}{\sqrt{2}}(|Hor. > +e^{i\delta\phi}|Vert. >)$ the probability of firing of detector 1 is 50 percent, the same for detector 2, and we can infer through which arm the photon passed (particle behaviour)
- If before reaching the polarisation beamsplitter in 9 the polarisation state of the light is $\frac{1}{2}((1+e^{i\delta\phi})|Hor. > +(1-e^{i\delta\phi})|Vert. >)$ the probability of firing of detector 1 is $cos^2(\delta\phi/2)$, and $sin^2(\delta\phi/2)$ for detector 2, and we cannot infer through which arm the photon passed (wave behaviour)
- Conclusion: the choice between wave and particle behaviour is realized in real time by the QRNG, at the fully end of the interferometer (48 meters long) in a time sufficiently short (smaller than 48 meters/c) so that Wheeler non-causal "necromancy" is guaranteed (no "distance loophole" here).

Modern perspective: the Q eraser.

Inteference quantum eraser



VDF: FILTRO A DENSITA' VARIABILE PER ATTENUARE L'INTENSITA' DEL RAGGIO LASER
WP: VETRO PIANO PARALLELO SU SUPPORTO ROTAZIONALE
BS: BEAMSPLITTER, DIVISORE DI FASCIO
LP: POLARIZZATORE LINEARE
PH: DIAFRAMMA CIRCOLARE CON DIAMETRO VARIABILE
L1,L2: LENTI DEL BEAM EXPANDER
T1, T2: TELECAMERE E RIVELATORI DI SINGOLO FOTONE

Modern perspective: the Q eraser.

- In a modern perspective, the wave-particle complementarity is put in relation with entanglement:

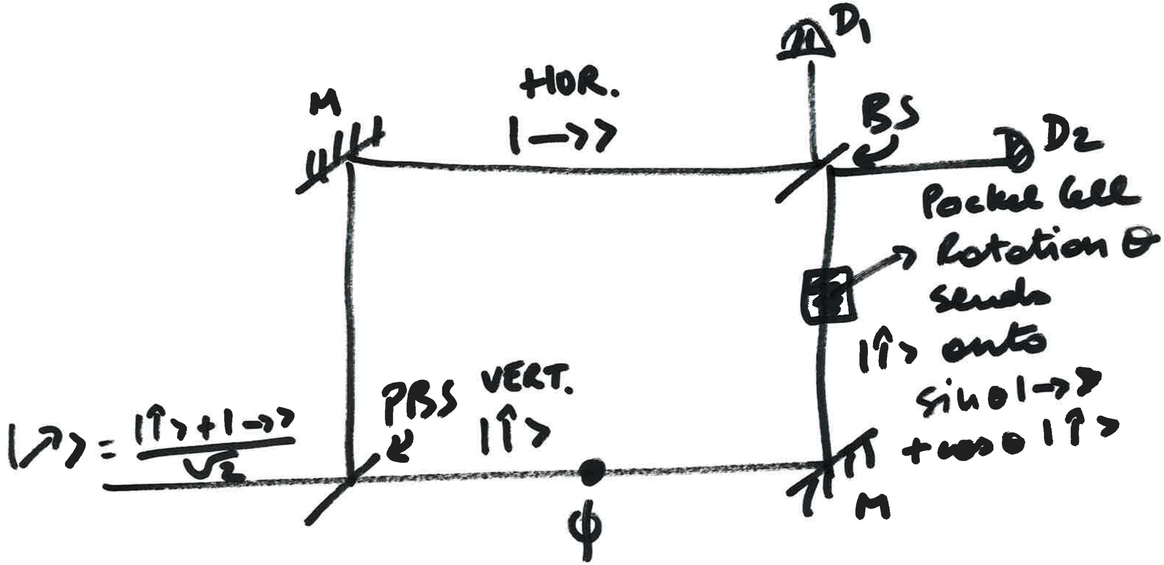
  the position of a quantum particle (A) passing through an interferometer is entangled with an extra-degree of freedom or ancilla (B).

- For instance (see figure next page), in a 2 arms M-Z interferometer, we entangle the position of a photon with its own polarisation.

- The state of the particle is thus prepared in such a way that its value before the output of the interferometer is equal to
  $\frac{1}{\sqrt{2}}(|up^A>|H^B>+e^{i\phi}|down^A>(sin\theta|H^B>+cos\theta|V^B>))$

Modern perspective: the Q eraser.

Q eraser. (my version of ...)



Modern perspective: the Q eraser.

- In matricial notation, let us adopt the following conventions:
  $| \, up^A >= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, | \, down^A >= \begin{pmatrix} 0 \\ 1 \end{pmatrix},$
  and $< \; up^A| = (1, 0), < \; down^A| = (0, 1).$

- One can show (see tutorials) by direct computation that when θ=0 the reduced density matrix[a] of the position degree of freedom (A) is
  $\rho^{reduced}_{up,down} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$
  this corresponds to a fully incoherent density matrix.

- Actually, when θ=0, entanglement is maximal and one can show easily that it is possible by measuring the polarisation at the end to guess which way the photon followed.

- This corresponds to a particle behaviour and interferences disappear then because the reduced density matrix assigned to the spatial position (up versus down) is fully incoherent.

[a] see tutorials and next chapters of this lessons-the A-reduced density matrix describes the effective state if we measure observables related to the position degree of freedom only-this is the case here: at the end of the Mach-Zehnder set up, the eigenstates associated to clicks in the two (+ and -) output detectors are $\frac{1}{\sqrt{2}}(| \, up^A > \pm | \, down^A >)|Hor.^B >$ and $\frac{1}{\sqrt{2}}(| \, up^A > \pm | \, down^A >)|Vert.^B >$. Polarisation plays no role here because the final beamsplitter does not distinguish them.

Modern perspective: the Q eraser.

- When θ=$\pi/2$, there is no entanglement at all (the state is factorisable in polarisation and space).
- Then no welcher-weg information is available.
- The A-reduced density matrix assigned to the spatial position then obeys $\rho_{up,down}^{reduced} = \begin{pmatrix} 1/2 & e^{i\phi}/2 \\ e^{-i\phi}/2 & 1/2 \end{pmatrix}$
- It describes a fully coherent state (pure state) and the visibility of the interferences is maximal (equal to 1). This corresponds to a wave behaviour.

Modern perspective: the Q eraser.

- For intermediate values of θ there is a continuum of situations where partial welcher-weg is available, ranging between the two extreme cases (particle and wave behaviour respectively).
- The reduced density matrix in the $|up^A>, |down^B>$ basis ranges from a pure, fully coherent state, to a fully incoherent state when θ varies from $\pi/2$ to 0.

Modern perspective: the Q eraser.

- Actually one gets $\rho_{up,down}^{reduced} = \begin{pmatrix} 1/2 & (e^{i\phi} sin\theta)/2 \\ (e^{-i\phi} sin\theta)/2 & 1/2 \end{pmatrix}$

- A possible measure[a] of the purity/coherence of $\rho_{up,down}^{reduced}$ is given by

  $2Tr.((\rho_{up,down}^{reduced})^2) - 1$;

  it is equal to $sin^2\theta$

- when θ varies from $\pi/2$ to 0, the visibility varies from 1 (wave behaviour, pure state, maximal visibility) to 0 (particle behaviour no interference, visibility=0).

- Let us define the visibility of the interferences as follows:

  Visibility=(Max probability in one detector-Min probability) normalised by (Max probability in one detector+Min probability).

- One can show that, in the present experiment, the visibility at the output of the interferometer is also equal to the purity.

[a] see next chapters and tutorials

Modern perspective: the Q eraser, some remarks and interpretations.
Remark 1.

- One is free to erase the welcher-weg information at the end of the interferometer by rotating the polarisation accordingly (see previous picture).
- This explains why this device is called the quantum eraser...

Modern perspective: the Q eraser, some remarks and interpretations.
Remark 2:

- There exists a complementarity relation between visibility of interferences and welcher-weg information;

  -it can be established independently of the state in which we prepare the system;

  - in the case of a pure state it can be put as we shall show soon in the form visibility + entanglement=1;

  -it can be generalized to arbitrary states (pure or mixed).

- This constitutes a modern version of Bohr's complementarity.

- It also constitutes as we shall see a fundamental aspect of quantum decoherence: when a system A is entangled with a system B, the purity of the reduced density matrix of A (B) decreases when the entanglement between A and B increases and vice versa.

Modern perspective: the Q eraser, some remarks and interpretations.
Remark 3:

- the "mechanical" interaction between measuring device and quantum system present in Einstein-Bohr's gedanken experiment is here replaced by entanglement between the system and its environment (or ancilla).
- The wave or particle nature of the quantum system can be controlled at will by varying the entanglement of the system with the rest of the world (ancilla).

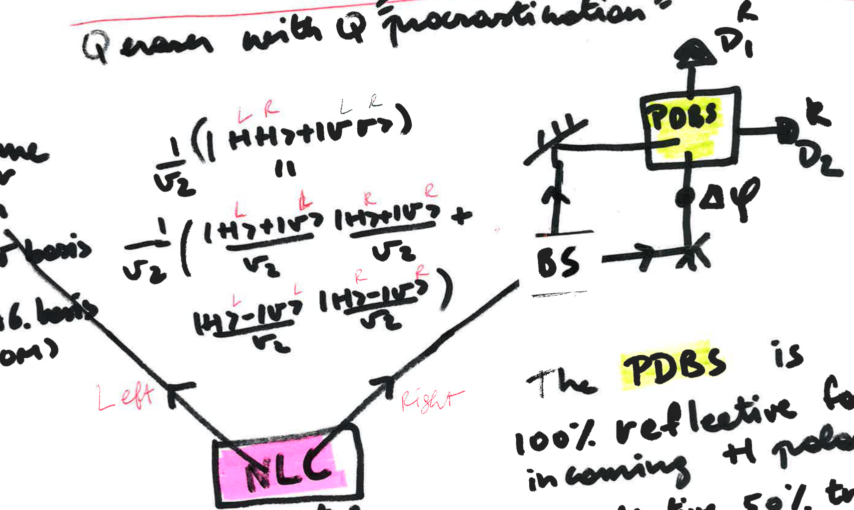
Quantum Eraser, an important experiment.

- **One step beyond into the quantum:**
- an important “recent” experiment:
- Kaiser, F., Coudreau, T., Milman, P., Ostrowsky, D. B. and Tanzilli, S. Entanglement-enabled delayed-choice experiment. Science 338, 637-640 (2012).

Entanglement-enabled delayed-choice experiment: Q eraser plus procrastination...

Entanglement-enabled delayed-choice exp.

Q eraser with Q "procrastination"



$$\frac{1}{\sqrt{2}}\left(|H^L H^R\rangle + |V^L V^R\rangle\right) = \frac{1}{\sqrt{2}}\left(\frac{|H\rangle+|V\rangle}{\sqrt{2}}^L \frac{|H\rangle+|V\rangle}{\sqrt{2}}^R + \frac{|H\rangle-|V\rangle}{\sqrt{2}}^L \frac{|H\rangle-|V\rangle}{\sqrt{2}}^R\right)$$

Measure Later either in H-V basis or in DIAG. basis (see ZOOM)

NLC: non linear crystal produces pairs of maximally entangled photons

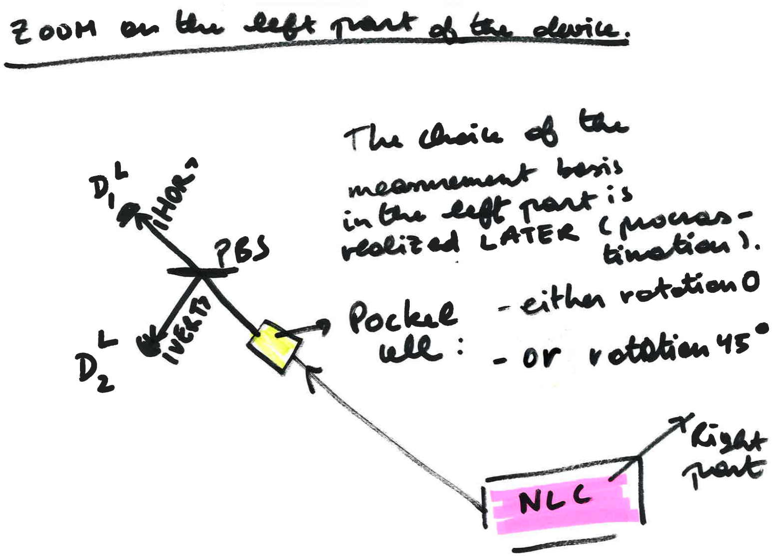
The PDBS is 100% reflective for incoming H polarisation
50% reflective 50% transmitter for incoming V polarisation.

Entanglement-enabled delayed-choice experiment: Q eraser plus procrastination...

- A source produces a pair of (maximally entangled in polarisation) photons.
- One of them is sent to the right part of the device where it passes through a beamsplitter.
- Thereafter the two branches are recombined in a PDBS (polarisation dependent beamsplitter) which is 100 percent reflective regarding incoming horizontal polarisations and acts as a 50-50 beamplitter on incoming vertical polarisations.
- This means that if the state is vertically polarised before entering the final interferometer, wave particle is realized and the visibility of interferences at the output is 1; on the contrary if the state is horizontally polarised before entering the interferometer, only one detector will click, revealing that the photon chose to follow the upper arm.
- If the polarisation state before entering the final interferometer takes an intermediate value between horizontal and vertical (for instance diagonal) the behaviour is neither a purely wave behaviour nor a purely particle behaviour but something in-between.

Entanglement-enabled delayed-choice experiment: Q eraser plus procrastination...

- As can be seen from the picture below, one is free to choose later, a long time after the right detectors clicked, to which basis the polarisation effectively belongs in the right arm by choosing to measure the left photon either in the hor-vert or in the diagonal polarisation basis.
- Postselecting the results in the right interferometer reveals the 3 behaviours made explicit before: wave, particle or in-between...
- This illustrates the idea of quantum procrastination: if we can do the job tomorrow, let us wait until tomorrow...



Decoherence. mathematical preamble.

- In order to properly describe decoherence, some preliminary concepts are necessary:

  -tensor product and entanglement

  -density matrix

  -reduced density matrix

  -measure of purity (coherence) of a density matrix

  -no signaling

Tensorial Product structure.
In modern books about the quantum theory it is often considered that the product structure deserves a postulate on its own. This structure is indeed essential: it describes how we couple quantum systems to each other, how we pass from one to many systems, and it radically differs from the classical picture. Among others entanglement is a by-product of the tensor product, which has no classical counterpart.

The **Tensor Product** is defined as follows:

-Let us consider a system A and a sytem B, respectively associated to the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$.

-then the system composed of the subsystems A and B is associated to the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, that is, the tensorial product of $\mathcal{H}_A$ and $\mathcal{H}_B$.

$\mathcal{H}_A \otimes \mathcal{H}_B$ is[a] defined as follows.

-it admits a basis that consists of ALL products of basis states of the A system and of the B system (of $\mathcal{H}_A$ and $\mathcal{H}_B$).

-their inner product is defined as follows:

$\forall|\psi_A>, |\phi_A> \in \mathcal{H}_A, \forall|\psi_B>, |\phi_B> \in \mathcal{H}_B$:

$< \psi_A| \otimes < \psi_B|| \phi_A > \otimes |\phi_B > = < \psi_A|\phi_A > \cdot < \psi_B|\phi_B >$.

[a] Mathematicians often use the symbol $\otimes$ but physicists rarely do so. For instance, the state $|+\rangle_Z^A \otimes |+\rangle_Z^B$ can be represented either by the symbol $|+\rangle_Z^A|+\rangle_Z^B$ or even $|+\rangle_Z|+\rangle_Z$ or yet $|++\rangle$, in situations where no ambiguity can occur.

Entanglement.

Linear combinations of factorisable states are not always factorisable; such states are called entangled states.

For instance Bell states are entangled

- Bell states are in 1-1 correspondence with Pauli spin operators:

  $\sigma_0 = |+\rangle_Z\langle+|_Z + |-\rangle_Z\langle-|_Z \leftrightarrow |B_0^0\rangle = \frac{1}{\sqrt{2}}(|+\rangle_Z^A \otimes |+\rangle_Z^B + |-\rangle_Z^A \otimes |-\rangle_Z^B)$

  $\sigma_x = |+\rangle_Z\langle-|_Z + |-\rangle_Z\langle+|_Z \leftrightarrow |B_1^0\rangle = \frac{1}{\sqrt{2}}(|+\rangle_Z^A \otimes |-\rangle_Z^B + |-\rangle_Z^A \otimes |+\rangle_Z^B)$

  $\sigma_y = i|+\rangle_Z\langle-|_Z - i|-\rangle_Z\langle+|_Z \leftrightarrow |B_0^1\rangle = \frac{1}{\sqrt{2}}(|+\rangle_Z^A \otimes |-\rangle_Z^B - |-\rangle_Z^A \otimes |+\rangle_Z^B)$

  $\sigma_z = |+\rangle_Z\langle+|_Z - |-\rangle_Z\langle-|_Z \leftrightarrow |B_0^1\rangle = \frac{1}{\sqrt{2}}(|+\rangle_Z^A \otimes |+\rangle_Z^B - |-\rangle_Z^A \otimes |-\rangle_Z^B)$

- Bell states are not factorizable; for instance if $|B_0^0\rangle$ would factorize then:

  $\alpha^A.\alpha^B$=$\beta^A.\beta^B$=$\sqrt{\frac{1}{2}}$ and $\alpha^A.\beta^B$= $\beta^A.\alpha^B$=0;

  Obviously such a system of equations has no solution (otherwise)

  $\alpha^A.\alpha^B.\beta^A.\beta^B$=$\sqrt{\frac{1}{2}}.\sqrt{\frac{1}{2}}$=$\frac{1}{2}$,

  and $\alpha^A.\beta^B.\beta^A.\alpha^B$=$0.0 = 0$ so that we run into a contradiction:

  $1/2 = 0$ (!?).

Entanglement and correlations.

- The product of two local observables O^A and O^B is defined as follows, through its action on factorizable states:
  $O^A \otimes O^B|\phi_A > \otimes|\phi_B >= (O^A|\phi_A >) \otimes (O^B|\phi_B >)$.

- It is easy to show that when the systems A and B are prepared in a factorisable state, the average of the product of local observables is the product of the local averages: $< O^A O^B >=< O^A >< O^B >$ (this is the so-called condition of statistical independence).

- Entangled states on the contrary ALWAYS exhibit correlations in well-chosen bases.

- The Bell state $|B_0^0\rangle = \frac{1}{\sqrt{2}}(|+\rangle_Z^A \otimes |+\rangle_Z^B + |-\rangle_Z^A \otimes |-\rangle_Z^B)$ for instance is such that both particles behave as perfect twin particles: they exhibit the same spin value when measured along an arbitrary direction (the same in the A and B regions) in the XZ plane on the Bloch sphere.

- The singlet state $|B_0^1\rangle = \frac{1}{\sqrt{2}}(|+\rangle_Z^A \otimes |-\rangle_Z^B - |-\rangle_Z^A \otimes |+\rangle_Z^B)$ is characterized by perfect anticorrelations: the outcomes for spin values measured along an arbitrary direction are always opposite to each other...

Density Matrix.

The density matrix was introduced by John von Neumann in order to solve the following problem :

... *Does there exist a convenient way to describe the state of a quantum system whenever its preparation consists of a statistical distribution of elements of the Hilbert space ? (the system is prepared in the state* $|\psi_i>$ *with probability* λ_i*:* $\lambda_i \in \mathbb{R}_+$*, and*$\Sigma_i \lambda_i = 1$*)...*

The average value of an observable O is then equal to the weighted sum of the average values corresponding to $|\psi_i>$ (weighted with the weights λ_i):

$< O >= \Sigma_i \lambda_i < O >_i = \Sigma_i \lambda_i < \psi_i | O | \psi_i >$.

It appears that $< \psi_i | O | \psi_i >$ is equal to the Trace[a] of a product of two operators: $< \psi_i | O | \psi_i >= Tr.O |\psi_i >< \psi_i|$, where $|\psi_i >< \psi_i|$ is the projector onto the state$|\psi_i >$.

The proof is left as an exercise (see tutorials).

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[a] Let us consider an arbitrary orthonormal basis $|\phi_j >, j = 1...D$ of $\mathcal{H}$, and A a linear operator, then, by definition $Tr.A = \Sigma_{j=1}^{D} < \phi_j |A| \phi_j >$. Actually, if A_{jk} is the matrix that represents A in the basis $|\phi_j >, j = 1...D$ ($A_{jk} =< \phi_j |A| \phi_k >$ and $A = \Sigma_{j,k=1}^{D} A_{j,k} |\phi_j >< \phi_k|$), then the trace is nothing else than the sum of the diagonal elements of this matrix.

Decoherence, mathematical preamble, density matrix.

Finally we find,

$< O >= \Sigma_i \lambda_i < O >_i = \Sigma_i \lambda_i < \psi_i|O|\psi_i >$

$= \Sigma_i \lambda_i Tr.(O|\psi_i >< \psi_i|) = Tr.(\Sigma_i \lambda_i |\psi_i >< \psi_i|O)$

$= Tr.(O\rho)$, where ρ is the **density matrix** of the system:

$\rho = \Sigma_i \lambda_i |\psi_i >< \psi_i|$.

In conclusion, once we have computed $\rho = \sum_i \lambda_i |\psi_i >< \psi_i|$ then we can compute all average values by computing a Trace:

$Tr.(O.\rho) = \sum < \psi_i|O|\psi_i > = < O >$.

Decoherence, mathematical preamble, Pure and Mixed states.

In the case that the state of the sytem is known without any ambiguity[a]:
$\lambda_i = \delta_{i,l}$ and $|\psi >= |\psi_l >$, then the density matrix is a one dimensional projector onto the state $|\psi_l >$:
$\rho = |\psi_l >< \psi_l|$.
In the generic case (mixed state), a density matrix is NOT always a projector but the following properties are ALWAYS valid:
Exercise:
if $\rho = \sum_i \lambda_i |\psi_i >< \psi_i|$ where the λ_i's are probabilities then

- ρ is a linear operator
- Trace(ρ)=1
- ρ is positive and self-adjoint
- ρ can thus be diagonalised in a certain basis ($|\tilde{e}_i >$) where it possesses a canonical expression of the form $\rho = \sum_i \tilde{\lambda}_i |\tilde{e}_i >< \tilde{e}_i|$where the $\tilde{\lambda}$'s are probabilities: $\sum_i \tilde{\lambda}_i = 1$ and $\tilde{\lambda}_i \in R^+, \forall i$

[a] Then it is said to be pure otherwise it is said to be mixed.

Decoherence, mathematical preamble, example of density matrix: thermalized state at temperature T.

Remark:

in statistical quantum mechanics, the state of the sytem when it is thermalized at temperature T is described by a density matrix:

$\rho = \Sigma_{E_i} p_i |E_i >< E_i|$,

where $p_i = \frac{exp(-\frac{E_i}{kT})}{\Sigma_i exp(-\frac{E_i}{kT})}$ and E_i represents eigenvalues of the Hamiltonian of the system.

Decoherence, mathematical preamble, reduced density matrix.
Let us assume that Alice and Bob's systems are prepared in the pure (but non necessarily factorizable) state $|\Psi\rangle^{AB} = \sum_{i,j=0}^{d-1} \alpha_{ij}|i\rangle^A \otimes |j\rangle^B$ (where $|i\rangle^A$ and $|j\rangle^B$ are states from orthonormalized reference bases) and that, say, Bob measures a local observable in the B region. Such an observable is represented by a local self-adjoint operator of the form $Id.^A \otimes O^B$ so that its average value is equal to

$$\begin{aligned}
&\sum_{i,j,i',j'=0}^{d-1} \alpha^*_{ij}\alpha_{i'j'}\langle i|^A \otimes \langle j|^B Id.^A \otimes O^B|i'\rangle^A \otimes |j'\rangle^B \\
= &\sum_{i,j,i',j'=0}^{d-1} \alpha^*_{ij}\alpha_{i'j'}\delta_{i,i'}\langle j|^B O^B|j'\rangle^B \\
= &\sum_{j,j'=0}^{d-1}\sum_{i=0}^{d-1} \alpha^*_{ij}\alpha_{ij'}\langle j|^B O^B|j'\rangle^B.
\end{aligned}$$

Reduced density matrix.

It is worth noting that the results of local Bob's measurements are the same as those that he would get if he would prepare his system in the state described by the effective or reduced density matrix $\rho^B = \sum_{j,j'=0}^{d-1} \sum_{i=0}^{d-1} \alpha_{ij}^* \alpha_{ij'} |j'\rangle^B \langle j|^B$.

Proof:

$Tr.(O^B.\rho^B) = Tr.(\sum_{j,j'=0}^{d-1} \sum_{i=0}^{d-1} \alpha_{ij}^* \alpha_{ij'} |j'\rangle^B \langle j|^B.O^B)$

$= \sum_{k=0}^{d-1} \langle k|^B (\sum_{j,j'=0}^{d-1} \sum_{i=0}^{d-1} \alpha_{ij}^* \alpha_{ij'} |j'\rangle^B \langle j|^B.O^B)|k\rangle^B$

$= \sum_{k=0}^{d-1} \delta k, j'^B (\sum_{j,j'=0}^{d-1} \sum_{i=0}^{d-1} \alpha_{ij}^* \alpha_{ij'} \langle j|^B.O^B)|k\rangle^B$

$= \sum_{j,j'=0}^{d-1} \sum_{i=0}^{d-1} \alpha_{ij}^* \alpha_{ij'} \langle j|^B O^B |j'\rangle^B$ in accordance with the average value computed before (previous page).

One can check (exercise) that ρ^B is well a trace 1, positive and self-adjoint operator as it must be.

Remark.

Formally, the reduced density matrix of Bob can be obtained by tracing out external degrees of freedom (in this case Alice's degrees of freedom):

$$
\begin{aligned}
Tr_A(|\Psi\rangle^{AB}\langle\Psi|^{AB}) &= Tr_A(\sum_{i,j=0}^{d-1}\alpha_{ij}|i\rangle^A\otimes|j\rangle^B\sum_{i',j'=0}^{d-1}\alpha^*_{i'j'}\langle i'|^A\otimes\langle j'|^B) \\
&= \sum_{k=0}^{d-1}\langle k|^A(\sum_{i,j=0}^{d-1}\alpha_{ij}|i\rangle^A\otimes|j\rangle^B\sum_{i',j'=0}^{d-1}\alpha^*_{i'j'}\langle i'|^A\otimes\langle j'|^B)|k\rangle^A \\
&= \sum_{k=0}^{d-1}(\sum_{i,j=0}^{d-1}\alpha_{ij}\delta_{k,i}|j\rangle^B\sum_{i',j'=0}^{d-1}\alpha^*_{i'j'}\delta_{k,i'}\langle j'|^B \\
&= \sum_{i,j,j'=0}^{d-1}\alpha_{ij}|j\rangle^B\alpha^*_{ij'}\langle j'|^B=\rho^B.
\end{aligned}
$$

Two measures of coherence (qubit case).

- Let us denote P_1 and P_2 the eigenvalues of the density matrix of a qubit; these are real positive numbers and their sum is equal to 1; they can be considered as the probabilities assigned to the first and second eigenstates of the density matrix.
- To quantify decoherence the most simple thing to do is to consider measure(decoherence)=1-measure(coherence)=$2(1 - Tr.(\rho^2))$=$4P_1P_2$.
- This measure is equal to 1 when $P_1 = P_2 = 1/2$ and to 0 when $P_1 = 0$ ($P_2 = 1$) or $P_1 = 1$ ($P_2 = 0$).
- To quantify coherence, we should then consider measure(coherence)=1-measure(decoherence)=$2Tr.(\rho^2) - 1$

Two measures of coherence (qubit case).
Alternative Measure of (de) coherence: Shannon-von Neumann entropy.

- An alternative measure of decoherence is provided by the Shannon-von Neumann entropy $-Tr.(\rho\, log_2\, \rho)$.

  Exercise: check that this measure is comprised between 0 and 1 and equal to 1 when $P_1 = P_2 = 1/2$ and to 0 when $P_1 = 0$ ($P_2 = 1$) or $P_1 = 1$ ($P_2 = 0$) (with P_1 and P_2 the eigenvalues of ρ).

  Remark: in dimension d the Shannon-von Neumann measure of decoherence is $-Tr.(\rho\, log_d\, \rho)$, it is again equal to 1 for fully incoherent mixtures and to 0 for pure states.

Mathematical preamble: No signaling theorem.

- As we have proven the local physics at, say, Bob's level is encapsulated in the reduced density matrix $\rho^B = \sum_{j,j'=0}^{d-1} \sum_{i=0}^{d-1} \alpha_{ij}^* \alpha_{ij'} |j'\rangle^B \langle j|^B$.
- The no signalling theorem expresses the fact that, when Alice and Bob are far away from each other, so that there is no direct interaction between Alice and Bob, Alice may not influence Bob's state IN AVERAGE.
- Even in the case of instantaneous action at a distance induced by the collapse process, Bob's reduced density matrix remains invariant after averaging over possible outcomes obtained by Alice when she measures a local (A) observable.

Mathematical preamble: No signaling theorem.

- In order to prove that Alice's action has no influence on the local properties of "Bob's" subsystem (we mean hereby the subsystem situated at Bob's side) we must establish the **INVARIANCE OF BOB'S REDUCED MATRIX** in two situations:

  A) Alice performs arbitrary measurements on "her" subsystem.

  B) Alice changes the local conditions of evolution at the level of "her" subsystem.

  More specifically one can show (appendix) that

  A) ρ^B remains, after Alice's measurement, the same as it was before.

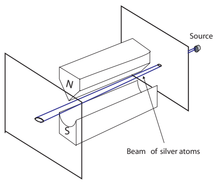
  B) If $H = H_A \otimes 1_B + 1_A \otimes H_B$, the evolution in time of ρ^B is the same as if $H_A = 0$, whatever the value of H_A.

Decoherence, example 1: Stern-Gerlach measure and spin decoherence.

- A silver atom is sent through a Stern-Gerlach device, and prepared in a pure spin state $|\Psi>_S = \alpha|+>_S + \beta|->_S$ where $|\pm>$ represents spin up (down) states along the Z direction.

- The initial spatial state, for instance a gaussian wave packet propagating along the X direction is denoted $|0>_E$; here the label E refers to the fact that we consider here the position degrees of freedom as an ancilla, an "external", "environmental" degree of freedom[a]

- The full state is thus initially a factorisable state
$$|\Psi>_{SE} = (\alpha|+>_S + \beta|->_S) \otimes |0>_E \, .$$



[a] This is the contrary of the quantum eraser where the photon spin played the role of the ancilla.

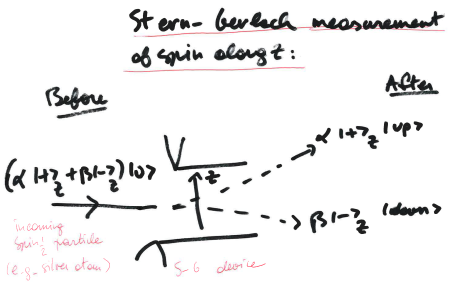
Decoherence, example 1: Stern-Gerlach measure and spin decoherence.

- After passing inside the S-G device, the full state is now an entangled state $|\Psi>_{SE}= (\alpha|+>_S \otimes |up>_E + \beta|->_S \otimes |down>_E)$, where for instance $|up>_E$ ($|down>_E$) represents a gaussian state moving upwards (downwards).

- The reduced density matrix of the spin degree of freedom undergoes thus the following transformation:

- Before the measurement: $\rho^{reduced}_{+,-} = \begin{pmatrix} |\alpha|^2 & \alpha^*\beta \\ \alpha\beta^* & |\beta|^2 \end{pmatrix}$ which represents a pure state.

- After the measurement, averaging over possible measurement outcomes: $\rho^{reduced}_{+,-} = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}$ which represents a mixed state.



Decoherence, example 2: localisation of a quantum bilocalised object after the scattering of one photon.

- Let us consider an object localised in two positions, up and down:
  $|\Psi>_S = \alpha|up>_S + \beta|down>_S$
- A photon is sent (see picture next page) in order to check whether the object is present in the up location[a].
  $|\Psi>_{SE} = (\alpha|up>_S + \beta|down>_S) \otimes |0>_E$.
- After passing in the up region, the state of the full system obeys
  $|\Psi>_{SE} =$
  $\alpha|up>_S \otimes |reflected>_E + \beta|down>_S \otimes |notreflected>_E$.

[a] This is the contrary of the quantum eraser where the photon played the role of the bilocalised system.

Scattering of a photon by a bilocalised object.

Decoherence, example 2: localisation of a quantum bilocalised object after the scattering of one photon.

- Before the measurement: $\rho^{reduced}_{up,down} = \begin{pmatrix} |\alpha|^2 & \alpha^*\beta \\ \alpha\beta^* & |\beta|^2 \end{pmatrix}$ which represents a pure state.
- After the measurement, averaging over possible measurement outcomes: $\rho^{reduced}_{up,down} = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}$ which represents a mixed state.

Decoherence, examples 1 and 2.

- As we see, the measurement process killed the off-diagonal elements (resonances) of the density matrix (expressed in the eigenbasis of the observable under measurement).
- Even in absence of a measurement apparatus, the entanglement with an ancilla kills the off-diagonal elements of the reduced density matrix, when it is expressed in diagonal form.
- **This is the so-called decoherence process.**

Four remarks about Decoherence.

- 1. The decoherence is present everywhere in nature: whenever a system interacts with its environment, they are likely to get entangled with each other[a] and the state of the system loses its purity (coherence).
- At the macro scale it is a very fast process because the coherence time is inversely proportional to the number of atoms in the system: deflection of one environmental photon by one atom suffices to localize a system as we have seen in example 2.

[a]One can show (T. Durt, Quantum Entanglement, Interaction, and the Classical Limit, Zeit. fur Nat. A 59, 425 (2004)) that if two systems interact without entangling arbitrary chosen initially factorisable states their interaction potential must be equal to 0. In other words, there is no interaction without entanglement. See also https://www.youtube.com/watch?v=CyFJgDgcEwc

Four remarks about Decoherence.

- 2. Decoherence succesfully explains for instance why we do not observe delocalised macroscopic states in everyday life, and why IBM, google and others did not reach supremacy yet with their quantum computer....
- Zeh, Zurek and others developed a formalism aimed at describing the evolution of a system in interaction with a noisy environment (this constitutes the theory of open systems).
- In particular the evolution of such systems is described via an irreversible in time[a] MASTER equation (often called the Lindblad equation).

[a]Irreversibility in time occurs here in the same way as in Boltzmann's approach: initially system and environment are not correlated (here entangled) which corresponds to low entropy; the entropy will spontaneously increase in time because the regions of large entropy are quite larger than the regions of low entropy (here we speak about the quantum entropy, measured for instance via the Shannon-von Neumann entropy).

Four remarks about Decoherence.

- 3. Some authors argue that decoherence makes it possible to get rid of the problematic collapse postulate and thus to solve the so-called measurement problem but this is not true.
- Decoherence does not kill the superpositions, it only kills their coherence. Decoherence only does not explain the so-called objectification process during which quantum potentialities get actualied.
- It does not explain what happens during individual measurement processes...
- If for instance we measure whether the photon passing through the up region in example 2 has been reflected or not in an individual process, this measurement will collapse the reduced state of the bilocalised system into the state $|up >_S$ (resp. $|down >_S$) with probability $|\alpha|^2$ (resp. $|\beta|^2$).

Four remarks about Decoherence.

- 3bis. Decoherence actually mimicks a measurement process during which we ignore what is the result of individual outcomes and are only interested in the average influence of the measurement.

- As a consequence of the no-signalling theorem, the reduced state of the system is the same in average, that we ignore or not the outcomes of the measurements performed on the ancilla, but this approach does not describe individual measurement processes; it only deals with the average effect of a large amount of measurements....

This explains why for instance Omnès, an influential promotor of the decoherence-based interpretation wrote in the preamble of one of his books the postulate 0: *Reality exists...*

Four remarks about Decoherence.

- 4. Typically, in a decoherence process, one can find a preferred basis $|\tilde{e}_i>_S$ which is determined by the type of environmental interaction that we consider such that an initial state $|\Psi(t=0)>_S \otimes|0>_E$ will evolve in time to an asymptotic state, reached after a time of the order of the (de)coherence time, of the form
  $\sum_i <\tilde{e}_i|_S|\Psi(t=0)>_S |\tilde{e}_i>_S \otimes|\tilde{f}_i>_E$
- The preferred basis $|\tilde{e}_i>$ is also said to result from an EIN (environment induced) superselection rule.
- From a technical point of view[a], the $|\tilde{e}_i>_S$ (resp.$|\tilde{f}_i>_E$) states belong to the orthonormal basis which diagonalises the reduced density matrix of the system S (resp. E) after interaction with the environment.

  For instance a Stern-Gerlach interaction wil select the spin basis as in example 1, while deflection by a photon as in example 2 or entanglement with polarisation as in the quantum eraser superselects the position basis.

[a] This is actually related to Schmidt biorthogonal decomposition, not detailed here.

Four remarks about Decoherence.

- 4bis. The decoherence process kills the coherences (off-diagonal elements of the reduced density matrix) in the preferred basis; it also "copies" the states of this basis in the environment.
- The states of the preferred basis are thus also called "quasi-classical pointer states" and they have the remarkable property that they interact with the environment without getting entangled with it.
- Superpositions of these pointer states however get entangled with the environment which kills their coherence...

Back to Bohr complementarity.

- In the case of pure states $|\Psi>_{SE}$ the Shannon-von Neumann entropy delivers at the same time a measure of decoherence of the reduced state and a measure of entanglement of the system with the rest of the world (environment).
- Coherence (purity) of the reduced state and entanglement between S and E are thus complementary in the Bohrian sense: the sum of their measures is equal to 1.
- This can be generalised to mixed states ρ_{SE}, but then the expression of complementarity is more complicate, we get then for instance that the sum of the welcher weg information and the visibility of interferences is bounded by a maximal value (see work of Englert, Scully, Bergou, Jaeger, Shimony and so on, see e.g. Jakobs and Bergou, Quantitative complementarity relations in bipartite systems, Optics Communications Volume 283, Issue 5, 1 March 2010, Pages 827-830).

Appendix. No signaling theorem: demonstration, part A.

If Alice performs a measurement in the $|i\rangle^A$ basis, she will project Bob's state onto a state proportional to the state $\sum_{j=0}^{d-1} \alpha_{ij}|j\rangle^B$ with probability $P_i = \sum_{j=0}^{d-1} |\alpha_{ij}|^2$. The projector on such a state, conveniently renormalized, is equal to $\frac{1}{P_i}\sum_{j,j'=0}^{d-1} \alpha_{ij}|j\rangle^B \alpha^*_{ij'}\langle j'|^B$ so that after averaging over all possible outcomes of Alice ($i : 0...d-1$), we get that $\langle O^B\rangle = \sum_{i=0}^{d-1} \frac{P_i}{P_i} \sum_{j,j'=0}^{d-1} \alpha_{ij}\alpha^*_{ij'}\langle j'|^B O^B|j\rangle^B$, equivalent to the average value that we derived in the absence of Alice's measurement.

Therefore if Alice performs her measurement at time t, and Bob measures O^B at time $t' = t + \epsilon$, then, $\langle O^B(t+\epsilon)\rangle = Tr.(\rho^B(t+\epsilon)O^B)$ before and after the measurement performed by Alice,

with $\rho^B(t-\epsilon) = \sum_{j,j'=0}^{d-1} \sum_{i=0}^{d-1} \alpha^*_{ij}(t-\epsilon)\alpha_{ij'}(t-\epsilon)|j'\rangle^B\langle j|^B = \rho^B(t+\epsilon)$, where ϵ represents a small time lapse, however longer than the (in principle arbitrarily short) duration of Alice's measurement.

Appendix. No signaling theorem: demonstration, part B.

Let us now explicitly compute the temporal evolution of

$\rho^B(t) = \sum_{j,j'=0}^{d-1} \sum_{i=0}^{d-1} \alpha^*_{ij}(t)\alpha_{ij'}(t)|j'\rangle^B\langle j|^B$,

in absence of Alice's measurement; in this case we get:

$i\hbar\frac{\partial}{\partial t}|\Psi\rangle_t^{AB} = i\hbar\frac{\partial}{\partial t}\sum_{i,j}\alpha_{ij}(t)|i\rangle^A \otimes |j\rangle^B$

$=H|\Psi\rangle_t^{AB} = (H_A \otimes 1_B + 1_A \otimes H_B)|\Psi\rangle_t^{AB}$

Henceforth $i\hbar\frac{\partial}{\partial t}\alpha_{ij}(t) = \sum_{kl} H_{ij,kl}\alpha_{kl}(t)$=$\sum_{kl}(H_A\otimes 1_B+1_A\otimes H_B)_{ij,kl}\alpha_{kl}(t)$;

Now, $(H_A \otimes 1_B)_{ij,kl} = (H_A)_{i,k} \cdot \delta_{j,l}$ while $(1_A \otimes H_B)_{ij,kl}$=$(H_B)_{j,l} \cdot \delta_{i,k}$.

Therefore, $\frac{\partial}{\partial t}\rho^B_{j',j}$=

$\sum_i \frac{\partial}{\partial t}(\alpha^*_{ij}(t)\alpha_{ij'}(t))$=$\sum_i(\frac{\partial}{\partial t}\alpha^*_{ij}(t))\alpha_{ij'}(t) + \sum_{i=0}^{d-1}\alpha^*_{ij}(t)\frac{\partial}{\partial t}(\alpha_{ij'}(t))$

=$\sum_i(\frac{-1}{i\hbar}\sum_{kl}((H_A)^*_{i,k} \cdot \delta_{j,l} + (H_B)^*_{j,l} \cdot \delta_{i,k})\alpha^*_{kl}(t))\alpha_{ij'}(t)$

$+\alpha^*_{ij}(t)\frac{1}{i\hbar}\sum_{kl}((H_A)_{i,k} \cdot \delta_{j',l} + (H_B)_{j',l} \cdot \delta_{i,k})\alpha_{kl}(t))$

Appendix.

No signaling theorem: demonstration, part B.

The contribution of Alice's Hamiltonian to the temporal evolution of $\rho^B_{j,j'}$ eads $\sum_i(\frac{-1}{i\hbar}\sum_{kl}(H_A)^*_{i,k}\cdot\delta_{j,l}\alpha^*_{k,l}(t)\alpha_{ij'}(t))+\frac{1}{i\hbar}(\sum_{kl}(H_A)_{i,k}\cdot\delta_{j',l}\alpha^*_{ij}(t)\alpha_{kl}(t)))$
$=\sum_i(\frac{-1}{i\hbar}\sum_k(H_A)_{k,i}\cdot\alpha^*_{k,j}(t)\alpha_{ij'}(t))+\frac{1}{i\hbar}(\sum_k(H_A)_{i,k}\cdot\alpha^*_{ij}(t)\alpha_{kj'}(t)))$
Now, denoting the mute index i by k and vice versa, we get $\sum_{i,k}(H_A)_{k,i}\cdot\alpha^*_{k,j}(t)\alpha_{ij'}(t)=\sum_{k,i}(H_A)_{i,k}\cdot\alpha^*_{i,j}(t)\alpha_{kj'}(t)$
so that Alice's contributions systematically cancel out. This ends the proof.

Appendix.

No signaling theorem: demonstration, part B: some remarks.

Remark 1:

- Of course when A and B are close to each other they may well influence each other by interacting through H_{AB}; signaling is then possible. However when they are far away from each other H_{AB} goes to zero (for instance, gravity and electro-magnetic interaction diminish with the distance).

Appendix.

No signaling theorem: demonstration, part B: some remarks.

Remark 2:

- When H_{AB}=0, the evolution of the reduced density matrix is only due to the influence of H_B; we get then $\frac{\partial}{\partial t}\rho^B_{j',j}$
  $=\sum_i(\frac{-1}{i\hbar}\sum_{kl}((H_B)^*_{j,l}\cdot\delta_{i,k})\alpha^*_{kl}(t))\alpha_{ij'}(t)+\alpha^*_{ij}(t)\frac{1}{i\hbar}\sum_{kl}((H_B)_{j',l}\cdot\delta_{i,k})\alpha_{kl}(t))$
  $=\sum_i(\frac{-1}{i\hbar}\sum_l((H_B)^*_{j,l})\alpha^*_{il}(t))\alpha_{ij'}(t)+\alpha^*_{ij}(t)\frac{1}{i\hbar}\sum_l((H_B)_{j',l})\alpha_{il}(t))$
  $=\frac{-1}{i\hbar}\sum_l(\sum_i\alpha_{ij'}(t)\alpha^*_{il}(t))\cdot(H_B)_{l,j}+\frac{1}{i\hbar}\sum_l(H_B)_{j',l}\cdot(\sum_i\alpha_{il}(t)\alpha^*_{ij}(t))$
  $=\frac{-1}{i\hbar}\sum_l\rho^B_{j',l}\cdot(H_B)_{l,j}+\frac{1}{i\hbar}\sum_l(H_B)_{j',l}\cdot\rho^B_{l,j}=\frac{1}{i\hbar}[H_B,\rho^B]_{j',j}$
- EXERCICE: check that this is exactly the same local evolution as for the density matrix of a fully isolated, disentangled, B system;

  in other words, if $\rho^B=\sum_k P_k|g_k><g_k|$, where the $|g_k>$ states are pure states of $\mathcal{H}_B$ which diagonalize ρ^B, then check by direct computation that if $i\hbar\frac{\partial}{\partial t}|g_k>=H_B|g_k>$, then

  $\frac{\partial}{\partial t}\rho^B=\frac{1}{i\hbar}[H_B,\rho^B]$ (this is the so-called Liouville-von Neumann equation).

Peyresq 2021.

PART 2.

Watching decoherence in real time: the experiment of Haroche and coworkers.

Aim of the experiment.

- In "normal"conditions, decoherence is a very fast process.
- For instance, as discussed in Schlosshauer's book "Decoherence And the Quantum-To-Classical Transition",

  the time taken for decoherence to occur for a grain of dust of size 10^{-3} **cm bilocated in two places separated by its own size (expressed in seconds)**

  -is equal to 1, if the environment only consisted of cosmic background radiation at 2,7 K;

  -at rooom temperature it would be 10^{-13};

  -it would be equal to 10^{-17} in the best laboratory vacuum,

  -and 10^{-29} in air at normal pressure.
- It is therefore very difficult to observe decoherence " in real time".
- Moreover it is very difficult to prepare macroscopic superpositions ("cat states").

Basic features of the experiment.
The aim of the experiment of Haroche's team is to observe in real time the achievement of the decoherence process.

- The system under interest here is a "kitten state" which is a superposition of two coherent states prepared inside a lossy cavity.
- Decoherence is here induced by the losses, and the evolution in time is described by a master equation of the Lindblad type.
- The scope of the experiment is to realize, at various times, Wigner tomography of the state of the light in the cavity, by using Rydberg atoms as probes.
- In fine, this experiment delivers a "movie" of the evolution of the state of the system, putting in evidence the gradual disappearance of the coherences between the two components of the kitten state, in a time notably faster than the decay time.

Content of the present chapter.

- We shall derive Lindblad equation nearly "from the scratch" in an input output formalism.
- We shall estimate the decay rate of coherences (off-diagonal terms of the reduced density matrix).
- We shall sketch the basic tricks making possible

  -1 to prepare kitten states.

  -2 to realize Wigner tomography of the state of light trapped in the cavity.

Derivation of the master equation for the state of light in a lossy cavity.

- Usually, to derive the master equation of a decoherent, open quantum system, one considers the coupling with the environment and performs a series of approximations, other approaches treat the irreversible in time interaction with the environment as a Monte-Carlo process (this is sketched in appendix).
- Here we present an in-out approach where we also keep track of the state of the environment (out system).
- Losses are treated as photons escaping from the cavity to the outside world.

Derivation of the master equation for the state of light in a lossy cavity.

- Our basic assumption is that the system coherently exchanges elementary excitations with its environment, that is, if at time $t = 0$,

  $$|\Psi_{S-E}(t=0)\rangle = |1_S\rangle \otimes |0_E\rangle$$

  then at time t,

  $$|\Psi_{S-E}(t)\rangle = \alpha(t)|1_S\rangle \otimes |0_E\rangle + \beta(t)|0_S\rangle \otimes |1_E\rangle$$

  where $\alpha(t)$ and $\beta(t)$ are properly normalized complex amplitudes (that is, they verify $|\alpha(t)|^2 + |\beta(t)|^2 = 1$) which we assume to know (*e.g.* through a Wigner-Weisskopf approach to the problem[a]).

- In principle the exact time-dependence of α and β can be derived once we know the interaction Hamiltonian H_{SE} between the system (S) and its environment (E).

[a]Debierre, V. La fonction d'onde du photon en principe et en pratique. Ph.D. thesis, Ecole Centrale Marseille (2015).

Derivation of the master equation for the state of light in a lossy cavity.

- Fock states evolve in time as follows:

$$\left|\Psi_{S-E}\left(t=0\right)\right\rangle = \left|n_S\right\rangle \otimes \left|0_E\right\rangle$$

$$\rightarrow \left|\Psi_{S-E}\left(t\right)\right\rangle = \sum_{m=0}^{n} \sqrt{\frac{n!}{m!\left(n-m\right)!}} \alpha^m\left(t\right)\beta^{n-m}\left(t\right)\left|m_S\right\rangle \otimes \left|\left(n-m\right)_E\right\rangle$$

where the square root comes from symmetrization[a] over all Fock states having m particles in the system and $n-m$ in the environment.

[a] In standard approaches, the normalisations result from properties of the photon creation-annihilation operators, as shown in V. Debierre, G. Demesy, T. Durt, A. Nicolet, B. Vial, et F. Zolla:"Absorption in quantum electrodynamics cavities in terms of a quantum jump operato", Physical Review A 90, 033806 (2014). However a wave function approach delivers exactly the same result as shown in T. Durt and V. Debierre, COHERENT STATES AND THE CLASSICAL-QUANTUM LIMIT CONSIDERED FROM THE POINT OF VIEW OF ENTANGLEMENT, Int J Mod Phys B 27, 1345014 (2013).

Derivation of the master equation for the state of light in a lossy cavity.

- In particular, if at time $t = 0$ the system is prepared in a coherent state and the environment is in the vacuum state, then the evolution yields coherent states for the environment too and the full state factorizes:

$$\left|\Psi_{S-E}\right\rangle(t=0) = \mathrm{e}^{-\frac{|\lambda|^2}{2}} \sum_{n=0}^{+\infty} \frac{\lambda^n}{\sqrt{n!}} |n_S\rangle \otimes |0_E\rangle$$

$$\rightarrow$$

$$\left|\Psi_{S-E}\right\rangle(t) =$$

$$\mathrm{e}^{-\frac{|\lambda|^2}{2}|\alpha(t)|^2} \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \frac{(\lambda\alpha(t))^m}{\sqrt{m!}} |m_S\rangle \otimes \mathrm{e}^{-\frac{|\lambda|}{2}|\beta(t)|^2} \frac{(\lambda\beta(t))^{n-m}}{\sqrt{(n-m)!}} \left|(n-m)_E\right\rangle$$

where we used the identity $|\alpha(t)|^2 + |\beta(t)|^2 = 1$.

Derivation of the master equation for the state of light in a lossy cavity.

- The state at time t can thus be rewritten as a product of coherent states:

$$|\mathbf{\Psi}_{S-E}(t)\rangle = \mathrm{e}^{-\frac{|\lambda|^2}{2}|\alpha(t)|^2}\sum_{k=0}^{+\infty}\frac{(\lambda\alpha(t))^k}{\sqrt{k!}}|k_S\rangle \otimes \mathrm{e}^{-\frac{|\lambda|^2}{2}|\beta(t)|^2}\sum_{l=0}^{+\infty}\frac{(\lambda\beta(t))^l}{\sqrt{l!}}|l_E\rangle.$$

- This establishes that coherent states of the system, in this regime, interact with the environment without getting entangled with it. They can thus be considered as "classical pointers" according to the criterion for classicality derived by Zurek in the framework of the quantum Darwinist approach[a].

[a] Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse? W. H. Zurek, Phys. Rev. D 24, 1516, 1981

Derivation of the master equation for the state of light in a lossy cavity.

- The basic ingredient of our derivation of the Linblad master equation in an IN-OUT formalism is to note that each damped coherent state of the form

$$|\Psi_S\rangle(t) = \mathrm{e}^{-\frac{|\lambda|^2}{2}|\alpha(t)|^2} \sum_{m=0}^{+\infty} \frac{(\lambda\alpha(t))^m}{\sqrt{m!}} |m_S\rangle,$$

  with $\alpha(t) = \mathrm{e}^{\mathrm{i}\omega t - \frac{\Gamma}{2}t}$, and $H = \hbar\omega a^\dagger a$, obeys the Lindblad equation (see appendix A for a more traditional derivation of Lindblad's equation) which is

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = \frac{1}{\mathrm{i}\hbar}[H, \rho_S(t)] + \frac{\Gamma}{2}\left[2a\rho_S(t)a^\dagger - a^\dagger a\rho_S(t) - \rho_S(t)a^\dagger a\right]$$

  This is proven in appendix B.

- In the same vein it can be shown that $|\Psi_S\rangle(t)\langle\Psi'_S|(t)$ with λ, λ' arbitrary complex numbers also obeys the Lindblad equation.

- Now , since coherent states constitute a basis of the Hilbert space[a], any density matrix must obey the Lindblad equation, by virtue of the linearity of the master equation. This ends our derivation.

[a] Coherent states are a basis in the sense that there exists a closure relation for such states. Rigorously, they form an overcomplete basis.

Decoherence of the kitten state.

- If at time $t = 0$ the state of the system is a coherent and symmetrical superposition of coherent states of opposite parity ("Schrödinger kitten"), while the environment is prepared in the vacuum state:

$$|\Psi_{S-E}(t=0)\rangle = \frac{\mathrm{e}^{-\frac{|\lambda|^2}{2}}}{\sqrt{2\left(1+\mathrm{e}^{-2|\lambda|^2}\right)}}\left(\sum_{n=0}^{+\infty}\frac{\lambda^n}{\sqrt{n!}}|n_S\rangle + \sum_{n=0}^{+\infty}\frac{(-\lambda)^n}{\sqrt{n!}}|n_S\rangle\right)\otimes|0_E\rangle$$

- then at time t the state of the full system will be given by

$$|\Psi_{S-E}(t)\rangle = \frac{\mathrm{e}^{-\frac{|\lambda|^2}{2}}}{\sqrt{2\left(1+\mathrm{e}^{-2|\lambda|^2}\right)}}\left[\sum_{k=0}^{+\infty}\frac{(\lambda\alpha(t))^k}{\sqrt{k!}}|k_S\rangle\otimes\sum_{m=0}^{+\infty}\frac{(\lambda\beta(t))^m}{\sqrt{m!}}|m_E\rangle\right.$$
$$\left. + \sum_{k=0}^{+\infty}\frac{(-\lambda\alpha(t))^k}{\sqrt{k!}}|k_S\rangle\otimes\sum_{m=0}^{+\infty}\frac{(-\lambda\beta(t))^m}{\sqrt{m!}}|m_E\rangle\right]$$

To prove this, note that the evolution is linear and make use of previous results regarding coherent states.

Decoherence of the kitten state.

- Let us estimate the reduced kitten state

$$\rho_S(t) = \mathrm{Tr}_E |\Psi_{S-E}(t)\rangle\langle\Psi_{S-E}(t)|.$$

- This reduced state, at time t, obeys

$$\begin{aligned}
\rho_S(t) = & \frac{\mathrm{e}^{-|\lambda\alpha(t)|^2}}{2\left(1+\mathrm{e}^{-2|\lambda|^2}\right))} \sum_{n=0}^{+\infty} \frac{(\lambda\alpha(t))^n}{\sqrt{n!}} |n_S\rangle \sum_{k=0}^{+\infty} \frac{(\lambda^*\alpha^*(t))^k}{\sqrt{k!}} \langle k_S| \\
& + \frac{\mathrm{e}^{-|\lambda\alpha(t)|^2}}{2\left(1+\mathrm{e}^{-2|\lambda|^2}\right)} \sum_{n=0}^{+\infty} \frac{(-\lambda\alpha(t))^n}{\sqrt{n!}} |n_S\rangle \sum_{k=0}^{+\infty} \frac{(-\lambda^*\alpha^*(t))^k}{\sqrt{k!}} \langle k_S| \\
& + \mathrm{e}^{-2|\lambda\beta(t)|^2} \frac{\mathrm{e}^{-|\lambda\alpha(t)|^2}}{2\left(1+\mathrm{e}^{-2|\lambda|^2}\right)} \sum_{n=0}^{+\infty} \frac{(\lambda\alpha(t))^n}{\sqrt{n!}} |n_S\rangle \sum_{k=0}^{+\infty} \frac{(-\lambda^*\alpha^*(t))^k}{\sqrt{k!}} \langle k_S| \\
& + \mathrm{e}^{-2|\lambda\beta(t)|^2} \frac{\mathrm{e}^{-|\lambda\alpha(t)|^2}}{2\left(1+\mathrm{e}^{-2|\lambda|^2}\right)} \sum_{n=0}^{+\infty} \frac{(-\lambda\alpha(t))^n}{\sqrt{n!}} |n_S\rangle \sum_{k=0}^{+\infty} \frac{(\lambda^*\alpha^*(t))^k}{\sqrt{k!}} \langle k_S|
\end{aligned}$$

Decoherence of the kitten state.

- The two first contributions represent an incoherent sum of the two components of the kittens.
- The two last terms represent the coherences (off-diagonal elements of the density matrix).
- Let us consider times of the order of $\frac{1}{|\lambda|^2\Gamma}$. The average number of photons at time $t = 0$ is of the order of $|\lambda|^2$ and we assume that it is large (5, 10, 100 or more).
- Then $\frac{1}{|\lambda|^2\Gamma}$ is small relatively to the decay time of one photon which is $\frac{1}{\Gamma}$
- However the coherences decrease like $\mathrm{e}^{-2|\lambda\beta(t)|^2}\mathrm{e}^{-|\lambda\alpha(t)|^2}$ and, as $\alpha(t) = \mathrm{e}^{\mathrm{i}\omega t-\frac{\Gamma}{2}t}$, it is easy to show that, for times of the order of $\frac{1}{|\lambda|^2\Gamma}$, the damping of the coherences is of the order of $\mathrm{e}^{-2|\lambda\Gamma t|^2}$.

Decoherence of the kitten state.

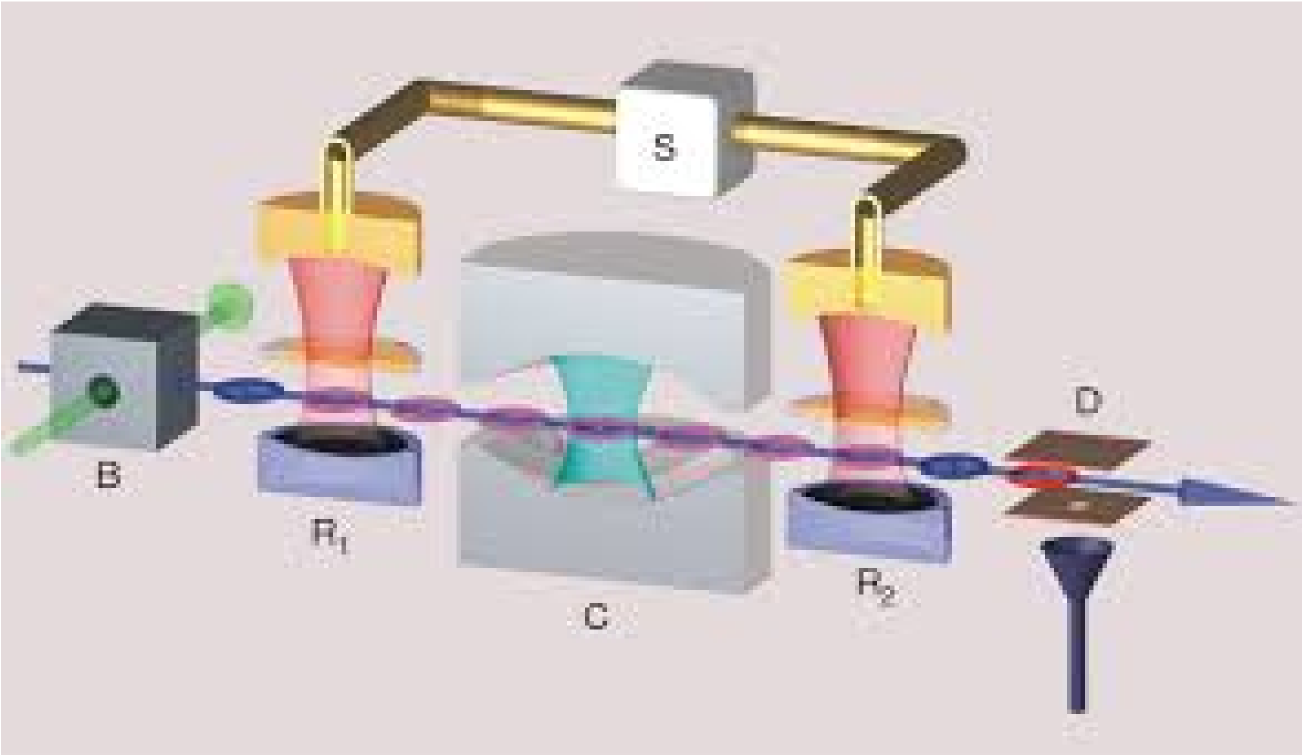
- Conclusion of previous page:

  this means that the decay of coherences is quite (5 times, 10 times, 100 times or more) faster than the decay of the number of photons, a feature of decoherence: the decoherence time is inversely proportional to the number of particles in the kitten, and to the distance between the two components of the kitten-state as well. Both parameters are here proportional to $|\lambda|^2$.

- To illustrate these concepts, we shall end this lesson by describing Haroche's experiment which made it possible to visualize decoherence in real time in a QED cavity.

Haroche experiment.

- Haroche Wigner tomography via entangled atoms passing through a QED cavity[a]

- 

[a] http://www.lkb.upmc.fr/cqed/wp-content/uploads/sites/14/2016/06/2009-LKB-AERES-StateReconstruction_low.pdf

Haroche experiment.

Preamble: Wigner distribution.

- Let us consider a quantum system of which the state-space is the Hilbert space $\mathcal{L}^2(\mathbb{R})$ (example: 1D harmonic oscillator).
- The Wigner operators $\hat{W}(x,p)$ behave as localisation operators in the phase-space associated to the system.
- They form an orthonormal basis of the liner operators acting on $\mathcal{L}^2(\mathbb{R})$.
- They are self-adjoint so that the amplitudes of the expansion of a density matrix $\hat{\rho}$ are real. These amplitudes are called the Wigner distribution of $\hat{\rho}$.
- It is sometimes called Wigner quasi-distribution because it can take real negative values.
- There is a one-to-one correspondence between $\hat{\rho}$ and its Wigner distribution.
- The Wigner distribution delivers a tomographic representation of a state $\hat{\rho}$.

Haroche experiment.

Preamble: Wigner distribution.

- The Wigner distribution can defined as
$w(x,p) = (2/h)\int dye^{-2ipy/\hbar} < x+y|\hat{\rho}|x-y >$.
- Making use of $e^{ix\hat{p}/\hbar}|y >= |y-x >$ and $< y|e^{ix\hat{p}/\hbar} =< y+x|$, it is easy to show that
$w(x,p) = (2/h)Tr.(e^{ixp/\hbar}e^{2(ip\hat{x}/\hbar)}e^{(2ix\hat{p}/\hbar)} \cdot \hat{Par.} \cdot \hat{\rho})$
$= (2/h)Tr.(e^{2(ip\hat{x}/\hbar+ix\hat{p}/\hbar)} \cdot \hat{Par.} \cdot \hat{\rho}) = Tr.\hat{W}(x,p) \cdot \hat{\rho}$,
where $\hat{Par.}$ represents the parity operator: $\hat{Par.}|y >= |-y >$.
- This means that the Wigner distribution is the trace of the product of $\hat{\rho}$ with the (twice) displaced parity operator.
- Let us consider a single mode, associated to light inside a cavity QED (actually, this is the minimal energy mode of the cavity).
- If we wish to perform tomography of the quantum state of light present in this mode, a possible strategy consists of directly measuring the average values of the Wigner operators; this has been done by the team of Serge Haroche (Nobel prize 2012).

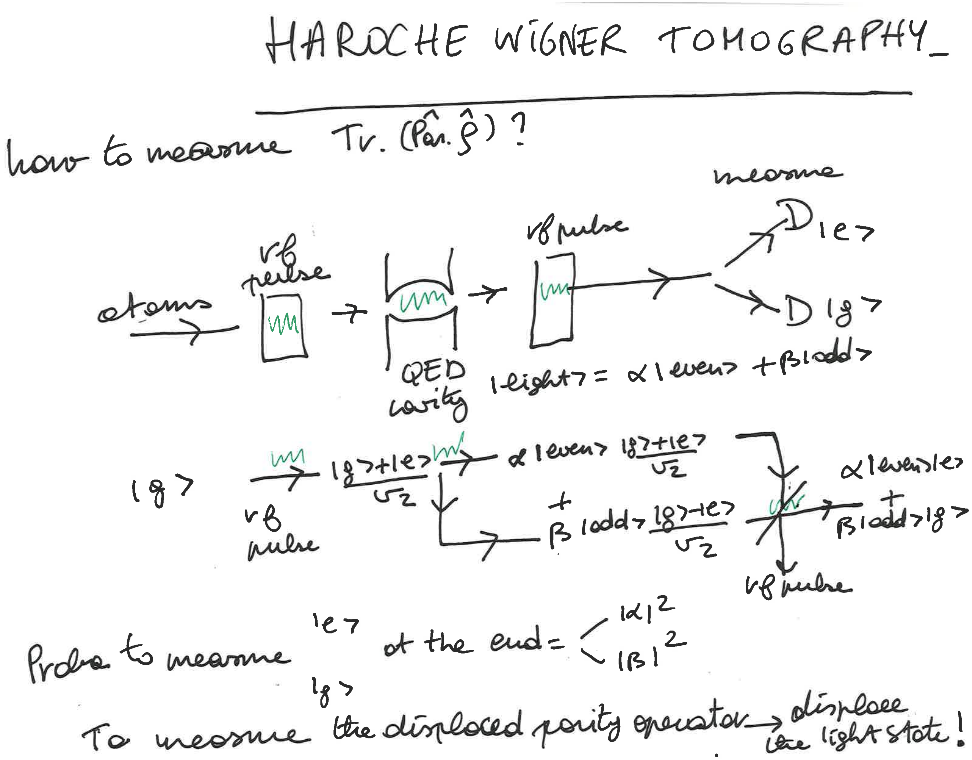
Haroche experiment:

preparation of the initial kitten state at time t=0.

- A kitten state is the projection of a coherent state onto the parity plus one eigenspace.
- To prepare a kitten state initially, prepare a coherent state inside the cavity by injecting light with a laser, then let pass one atome inside the cavity, the final mesurement will project the coherent state on a kitten state with probability one halve.
- This is illustrated on the figure next page, where we see that if the state of light before the passage of one atom (prepared in the ground state) is denoted $|\Psi>_L$, the state at the output of the device will be
$(1/\sqrt{2})(|\Psi^+>_L|e>_A+|\Psi^->_L|g>_A)$,
where $|\Psi^+>_\pm=(1/2)(1\pm Par.)|\Psi>_L$, while $|g/e>_A$ denote the atomic excited/ground states.
- Note that $(1/2)(1+Par.)+(1/2)(1-Par.)=1$, and $((1/2)(1\pm Par.))^2=(1/2)(1\pm Par.)$ so that $(1/2)(1\pm Par.)$ can be considered as the projector onto eigenstates of the parity operator for eigenvalues ±1...

Haroche experiment.

- Haroche Wigner tomography via entangled atoms passing through a QED cavity.



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Haroche experiment:

Wigner tomography of the kitten state at time t.

- To measure the displaced parity operator at time t of a state initially prepared in a kitten state, displace the light state by injecting laser light at time $t - \epsilon$, then let pass one atome inside the cavity at time $t + \epsilon$, the final mesurement will deliver two outcomes excited or ground) which are in one to one correspondence with the eigenstates of the parity operator associated to even (+1) and odd (-1) parity.
- Repeating many times the whole process and measuring the frequencies of occurences of excited (parity +1) and ground states (parity -1) at the output, we have access to the average value of the parity operator.
- This delivers a picture of the quantum state of the kitten at time t.
- TIf we make use of 100 atoms to estimate the average value of the parity operator, then, to make a picture with 50 times 50 Wigner amplitudes requires 2 times 50 times 50 times 100 atoms.

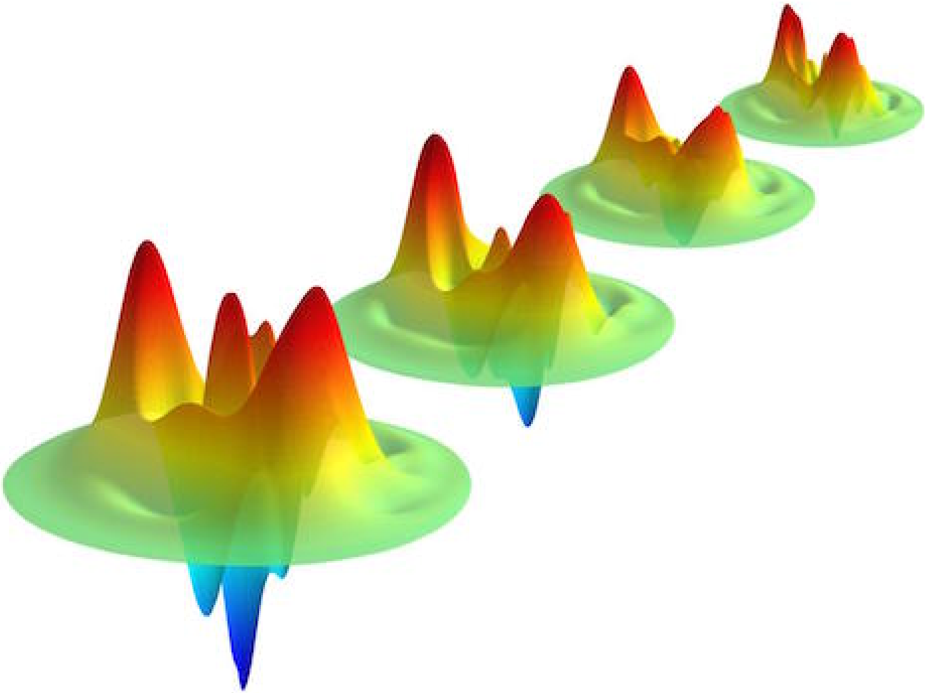
Haroche experiment:

Wigner tomography of the kitten state: the movie.

- To make a movie of the quantum state of light, repeat the previous process for several times t.
- Example: to obtain a movie with pictures taken at 100 different times requires 100 times 2 times 50 times 50 times 100 atoms=5.10^7 atoms.
- For an initial kitten state with 10 photons, the decoherence time is of the order of 130 miliseconds (lifetime of one photon in the cavity) divided by 10.
- The movie can thus be realized in more or less 5.10^5 seconds...less than 6 days..

Haroche experiment.

- Haroche Wigner tomography: movie of decoherence of a cat state in a lossy cavity QED[a].



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[a] http://www.lkb.upmc.fr/cqed/non-local-quantum-states/

Appendix A: Monte-Carlo derivation of the master equation for the state of light in a lossy cavity.

Traditional, Monte-Carlo derivation of the Linblad master equation.

- Decoherence, dissipation and so on are characterized by a unique factor, Γ, the loss-rate of the system.

- Let us assume that at time t the system is a n photon-Fock state (more generally an energy state that consists of n elementary excitations). During the time interval $[t, t+\mathrm{d}t]$, one elementary excitation of the system is dissipated in the environment with probability $\Gamma n \, \mathrm{d}t$, in which case the state at time t should be replaced by the properly normalised $(n-1)$ photon Fock state $a\Psi_S(t)/\sqrt{n}$ at time $t + \mathrm{d}t$. Otherwise (and this happens with probability $(1 - \Gamma n \, \mathrm{d}t)$), no excitation is released and the state of the system at time $t + \mathrm{d}t$ is equal to $\sqrt{1 - \Gamma n \, \mathrm{d}t} \, (\Psi_S(t) + H \, (1/\mathrm{i}\hbar) \, \mathrm{d}t \, \Psi_S(t))$ (which is to be normalized in the form in which it was written).

Traditional, Monte-Carlo derivation of the Linblad master equation.
Putting everything together, on average, $\rho\left(t\right)$ thus evolves during the time interval $[t, t+\mathrm{d}t]$ to the Lindblad equation

$$\rho(t+\mathrm{d}t) = \rho\left(t\right) + \frac{\Gamma n\,\mathrm{d}t\,a\rho\left(t\right)a^{\dagger}}{n} + \mathrm{d}t\frac{1}{\mathrm{i}\hbar}\left[H, \rho\left(t\right)\right] - \frac{\Gamma n\,\mathrm{d}t}{2}\rho\left(t\right),$$

Equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho\left(t\right) = \frac{1}{\mathrm{i}\hbar}\left[H, \rho\left(t\right)\right] + \frac{\Gamma}{2}\left[2a\rho_S\left(t\right)a^{\dagger} - a^{\dagger}a\rho\left(t\right) - \rho\left(t\right)a^{\dagger}a\right]$$

Appendix B: Derivation of the master equation for the state of light in a lossy cavity in the IN-OUt approach: complements. Proof that reduced pointer states obey Lindblad equation.

- We prove here that coherent states are solutions to the Lindblad master equation of a lossy cavity. We begin by writing the system's reduced density matrix for coherent statesexponentially decaying coherent states. It is straightforward to see that the density matrix of such a damped coherent state reads

$$\rho_S\left(t\right) = \mathrm{e}^{-\left|\lambda\alpha(t)\right|^2}\sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{\left(\lambda\alpha\left(t\right)\right)^k}{\sqrt{k!}}\frac{\left(\lambda^*\alpha^*\left(t\right)\right)^n}{\sqrt{n!}}|k_S\rangle\langle n_S|.$$

  Then we proceed to differentiate it with respect to time:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\rho_S\left(t\right)\\ &= \mathrm{e}^{-\left|\lambda\alpha(t)\right|^2}\left[-\left|\lambda\right|^2\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha\left(t\right)\right|^2\sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{\left(\lambda\alpha\left(t\right)\right)^k}{\sqrt{k!}}\frac{\left(\lambda^*\alpha^*\left(t\right)\right)^n}{\sqrt{n!}}|k_S\rangle\langle n_S| + \right.\\ &\left.\sum_{k,n=1}^{+\infty}\left(k\lambda\frac{\mathrm{d}\alpha}{\mathrm{d}t}\lambda^*\alpha^*\left(t\right) + n\lambda^*\frac{\mathrm{d}\alpha^*}{\mathrm{d}t}\lambda\alpha^*\left(t\right)\right)\frac{\left(\lambda\alpha\left(t\right)\right)^{k-1}}{\sqrt{k!}}\frac{\left(\lambda^*\alpha^*\left(t\right)\right)^{n-1}}{\sqrt{n!}}|k_S\rangle\langle n_S|\right]\end{aligned}$$

Proof that reduced pointer states obey Lindblad equation.

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t)$$
$$= \mathrm{e}^{-|\lambda\alpha(t)|^2}\sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty}\left[-|\lambda|^2\frac{\mathrm{d}}{\mathrm{d}t}|\alpha(t)|^2+\left(\frac{k}{\alpha(t)}\frac{\mathrm{d}\alpha}{\mathrm{d}t}+\frac{n}{\alpha^*(t)}\frac{\mathrm{d}\alpha^*}{\mathrm{d}t}\right)\right]$$
$$\frac{(\lambda\alpha(t))^k}{\sqrt{k!}}\frac{(\lambda^*\alpha^*(t))^n}{\sqrt{n!}}|k_S\rangle\langle n_S|.$$

- We now use the Wigner-Weisskopf expression for α:

$$\alpha(t)=\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\frac{1}{2}\Gamma t}$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t)=\mathrm{e}^{-|\lambda\alpha(t)|^2}\sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty}\left[\Gamma\left(|\lambda|^2|\alpha(t)|^2-\frac{1}{2}(k+n)\right)-\mathrm{i}\omega(k-n)\right]$$
$$\frac{(\lambda\alpha(t))^k}{\sqrt{k!}}\frac{(\lambda^*\alpha^*(t))^n}{\sqrt{n!}}|k_S\rangle\langle n_S|.$$

Proof that reduced pointer states obey Lindblad equation.

- Now, recalling that $H = \hbar\omega a^\dagger a$:

$$\frac{1}{\mathrm{i}\hbar}\left[H,\rho_S\left(t\right)\right] = \frac{1}{\mathrm{i}}\mathrm{e}^{-\left|\lambda\alpha(t)\right|^2}\sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{\left(\lambda\alpha\left(t\right)\right)^k}{\sqrt{k!}}\frac{\left(\lambda^*\alpha^*\left(t\right)\right)^n}{\sqrt{n!}}\left[\omega a^\dagger a,|k_S\rangle\langle n_S|\right]$$

$$= -\mathrm{i}\,\mathrm{e}^{-\left|\lambda\alpha(t)\right|^2}\sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{\left(\lambda\alpha\left(t\right)\right)^k}{\sqrt{k!}}\frac{\left(\lambda^*\alpha^*\left(t\right)\right)^n}{\sqrt{n!}}\omega\left(k-n\right)|k_S\rangle\langle n_S|,$$

while

$$\frac{\Gamma}{2}\left[2a\rho_S\left(t\right)a^\dagger - a^\dagger a\rho_S\left(t\right) - \rho_S\left(t\right)a^\dagger a\right]$$

$$= \mathrm{e}^{-\left|\lambda\alpha(t)\right|^2}\left[2\sum_{k=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{\left(\lambda\alpha\left(t\right)\right)^k}{\sqrt{k!}}\frac{\left(\lambda^*\alpha^*\left(t\right)\right)^n}{\sqrt{n!}}\sqrt{kn}|\left(k-1\right)_S\rangle\langle\left(n-1\right)_S|\right.$$

$$\left.-\sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty}\left[\frac{\left(\lambda\alpha\left(t\right)\right)^k}{\sqrt{k!}}\frac{\left(\lambda^*\alpha^*\left(t\right)\right)^n}{\sqrt{n!}}k - \frac{\left(\lambda\alpha\left(t\right)\right)^k}{\sqrt{k!}}\frac{\left(\lambda^*\alpha^*\left(t\right)\right)^n}{\sqrt{n!}}n\right]|k_S\rangle\langle n_S|\right] =$$

$$\mathrm{e}^{-\left|\lambda\alpha(t)\right|^2}\sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty}\left[2\left|\lambda\right|^2\left|\alpha\left(t\right)\right|^2-\left(k+n\right)\right]\frac{\left(\lambda\alpha\left(t\right)\right)^k}{\sqrt{k!}}\frac{\left(\lambda^*\alpha^*\left(t\right)\right)^n}{\sqrt{n!}}n|k_S\rangle\langle n_S|.$$

Proof that reduced pointer states obey Lindblad equation.

- Putting everything together, we immediately see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S\left(t\right) = \frac{1}{\mathrm{i}\hbar}\left[H, \rho_S\left(t\right)\right] + \frac{\Gamma}{2}\left[2a\rho_S\left(t\right)a^\dagger - a^\dagger a\rho_S\left(t\right) - \rho_S\left(t\right)a^\dagger a\right]$$

  is verified, which ends the proof.